

Para-Blaschke isoparametric spacelike hypersurfaces in Lorentzian space forms

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Abstract

Let M^n be an n -dimensional umbilic-free hypersurface in the $(n+1)$ -dimensional Lorentzian space form $M_1^{n+1}(c)$. Three basic invariants of M^n under the conformal transformation group of $M_1^{n+1}(c)$ are a 1-form C , called conformal 1-form, a symmetric $(0, 2)$ tensor B , called conformal second fundamental form, and a symmetric $(0, 2)$ tensor A , called Blaschke tensor. The so-called para-Blaschke tensor $D^\lambda = A + \lambda B$, the linear combination of A and B , is still a symmetric $(0, 2)$ tensor. A spacelike hypersurface is called a para-Blaschke isoparametric spacelike hypersurface, if the conformal 1-form vanishes and the eigenvalues of the para-Blaschke tensor are constant. In this paper, we classify the para-Blaschke isoparametric spacelike hypersurfaces under the conformal group of $M_1^{n+1}(c)$.

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1 Introduction

Recently the Möbius geometry of submanifolds in Riemannian space forms has been studied extensively and a lot of interesting results have been obtained. Especially, Many

special hypersurfaces were classified under Möbius transformation group (for example, [2],[3],[4],[5],[6],[9],[10],[13],[14]). As its parallel generalization, the conformal geometry of submanifolds in Lorentzian space forms is another important branch of conformal geometry, but there are less results in Lorentzian space forms than in Riemannian space forms. In this paper, we study the para-Blaschke isoparametric spacelike hypersurfaces in Lorentzian space forms.

Let \mathbb{R}_s^{n+2} be the real vector space \mathbb{R}^{n+2} with the Lorentzian product $\langle \cdot, \cdot \rangle_s$ given by

$$\langle X, Y \rangle_s = - \sum_{i=1}^s x_i y_i + \sum_{j=s+1}^{n+2} x_j y_j.$$

For any $a > 0$, the standard sphere $\mathbb{S}^{n+1}(a)$, the hyperbolic space $\mathbb{H}^{n+1}(-a)$, the de sitter space $\mathbb{S}_1^{n+1}(a)$ and the anti-de sitter space $\mathbb{H}_1^{n+1}(-a)$ are defined by

$$\begin{aligned} \mathbb{S}^{n+1}(a) &= \{x \in \mathbb{R}^{n+2} | x \cdot x = a^2\}, \quad \mathbb{H}^{n+1}(-a) = \{x \in \mathbb{R}_1^{n+2} | \langle x, x \rangle_1 = -a^2\}, \\ \mathbb{S}_1^{n+1}(a) &= \{x \in \mathbb{R}_1^{n+2} | \langle x, x \rangle_1 = a^2\}, \quad \mathbb{H}_1^{n+1}(-a) = \{x \in \mathbb{R}_2^{n+2} | \langle x, x \rangle_2 = -a^2\}. \end{aligned}$$

Let $M_1^{n+1}(c)$ be a Lorentzian space form. When $c = 0$, $M_1^{n+1}(c) = \mathbb{R}_1^{n+1}$. When $c = 1$, $M_1^{n+1}(c) = \mathbb{S}_1^{n+1}(1)$. When $c = -1$, $M_1^{n+1}(c) = \mathbb{H}_1^{n+1}(-1)$.

For Lorentzian space forms $M_1^{n+1}(c)$, there exists a united conformal compactification \mathbb{Q}_1^{n+1} , which is the projectivized light cone in $\mathbb{R}P^{n+2}$ induced from \mathbb{R}_2^{n+3} . Using the conformal compactification \mathbb{Q}_1^{n+1} , we study the conformal geometry of spacelike hypersurfaces in $M_1^{n+1}(c)$. We define the conformal metric g and the conformal second fundamental form B on an umbilic-free spacelike hypersurface, which determine the spacelike hypersurface up to a conformal transformation of $M_1^{n+1}(c)$. Another two conformal invariants are the conformal 1-form C and the Blaschke tensor A (see Sect.2).

Since A and B are symmetric $(0, 2)$ -tensor, their eigenvalues are real. We define two kind of special spacelike hypersurfaces: the conformal isoparametric spacelike hypersurfaces and the Blaschke isoparametric spacelike hypersurfaces. A spacelike hypersurface is called a conformal isoparametric spacelike hypersurface, if it satisfies two conditions: (1) $C = 0$, (2) all the eigenvalues of B are constant. Similarly, we define the Blaschke isoparametric spacelike hypersurface by another symmetric tensor, the Blaschke tensor A . The para-Blaschke tensor defined by $D^\lambda := A + \lambda B$ for some constant λ . Clearly the

para-Blaschke tensor is still a symmetric $(0, 2)$ tensor, thus its eigenvalues are real. Using the para-Blaschke tensor, we can define similarly the para-Blaschke isoparametric spacelike hypersurface.

Recently, some interesting results on the spacelike hypersurfaces with some special conformal invariants are obtained. C. X. Nie et al. classified the spacelike hypersurfaces with parallel conformal second fundamental form in [17], and classified the Blaschke isoparametric spacelike hypersurfaces with two distinct principal curvatures in [16]. X. X. Li et al. classified the spacelike hypersurfaces with parallel Blaschke tensor in [11] and the spacelike hypersurfaces with parallel para-Blaschke tensor in [12]. T.Z. Li and C.X. Nie classified completely the conformal isoparametric spacelike hypersurfaces in [8]. Clearly if the para-Blaschke tensor of a spacelike hypersurface is parallel, then the spacelike hypersurface is para-Blaschke isoparametric. In this paper, we prove that a para-Blaschke isoparametric spacelike hypersurface is a conformal isoparametric spacelike hypersurface provided that the para-Blaschke tensor has more than two distinct eigenvalues. Simultaneously, we classify completely the para-Blaschke isoparametric spacelike hypersurfaces. Our main theorems are as follows.

Theorem 1.1. *Let $x : M^n \rightarrow M_1^{n+1}(c)$, $n \geq 2$, be an umbilic-free spacelike hypersurface in an $(n + 1)$ -dimensional Lorentzian space form $M_1^{n+1}(c)$. We assume that the conformal 1-form of x vanishes. Then we have*

(1), *If the spacelike hypersurface is a conformal isoparametric spacelike hypersurface, then the spacelike hypersurface is also a para-Blaschke isoparametric spacelike hypersurface.*

(2), *If the spacelike hypersurface is a para-Blaschke isoparametric spacelike hypersurface and the number of the distinct eigenvalues of the para-Blaschke tensor D^λ is more than two, then the spacelike hypersurface is also a conformal isoparametric spacelike hypersurface.*

Theorem 1.2. *Let $x : M^n \rightarrow M_1^{n+1}(c)$, $n \geq 2$, be an umbilic-free spacelike hypersurface in an $(n + 1)$ -dimensional Lorentzian space form $M_1^{n+1}(c)$. If the hypersurface is para-Blaschke isoparametric, then x is locally conformal equivalent to one of the following hypersurfaces:*

(1), *the spacelike hypersurfaces with constant mean curvature and constant scalar cur-*

nature in $M_1^{n+1}(c)$;

(2), $\mathbb{S}^k(\sqrt{a^2+1}) \times \mathbb{H}^{n-k}(-a) \subset \mathbb{S}_1^{n+1}(1)$, $a > 0$, $1 \leq k \leq n-1$;

(3), $\mathbb{H}^k(-a) \times \mathbb{H}^{n-k}(-\sqrt{1-a^2}) \subset \mathbb{H}_1^{n+1}(-1)$, $0 < a < 1$, $1 \leq k \leq n-1$;

(4), $\mathbb{H}^k(-a) \times \mathbb{R}^{n-k} \subset \mathbb{R}_1^{n+1}$, $a > 0$, $0 \leq k \leq n-1$;

(5), $x : \mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \rightarrow \mathbb{R}_1^{n+1}$, defined by

$$x(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2-1})$, $u'' \in \mathbb{S}^p(a)$, $u''' \in \mathbb{R}^{n-p-q-1}$, $a > 1$;

(6) the spacelike hypersurfaces defined by Example 3.5 (see Sect.3);

(7) the spacelike hypersurfaces defined by Example 3.6 (see Sect.3).

When $\lambda = 0$, $D^\lambda = A$. Theorem 1.2 implies that the conformal isoparametric space-like hypersurfaces and the Blaschke isoparametric spacelike hypersurfaces are almost equivalent. Therefore from the results in [8], we have the following results.

Corollary 1.1. *Let $x : M^n \rightarrow M_1^{n+1}(c)$, $n \geq 2$, be an umbilic-free spacelike hypersurface in the $(n+1)$ -dimensional Lorentzian space form $M_1^{n+1}(c)$ with r distinct eigenvalues of the Blaschke tensor. If the hypersurface is Blaschke isoparametric and $r \geq 3$, then $r = 3$ and x is locally conformal equivalent to the following spacelike hypersurface:*

$$x : \mathbb{H}^q(-\sqrt{a^2-1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \rightarrow \mathbb{R}_1^{n+1},$$

defined by $x(u', u'', t, u''') = (tu', tu'', u''')$, where $u' \in \mathbb{H}^q(-\sqrt{a^2-1})$, $u'' \in \mathbb{S}^p(a)$, $u''' \in \mathbb{R}^{n-p-q-1}$, $a > 1$.

This paper is organized as follows. In section 2, we study the conformal geometry of spacelike hypersurfaces in $M_1^{n+1}(c)$. In section 3, we give some examples of special spacelike hypersurfaces. In section 4 we give the proof of our main theorems.

2 Conformal geometry of spacelike Hypersurfaces

In this section, following Wang's idea in paper [19], we define some conformal invariants on a spacelike hypersurface and give a congruent theorem of the spacelike hypersurfaces under the conformal group of $M_1^{n+1}(c)$.

We denote by C^{n+2} the cone in \mathbb{R}_2^{n+3} and by \mathbb{Q}_1^{n+1} the conformal compactification space in $\mathbb{R}P^{n+2}$,

$$C^{n+2} = \{X \in \mathbb{R}_2^{n+3} | \langle X, X \rangle_2 = 0, X \neq 0\},$$

$$\mathbb{Q}_1^{n+1} = \{[X] \in \mathbb{R}P^{n+2} | \langle X, X \rangle_2 = 0\}.$$

Let $O(n+3, 2)$ be the Lorentzian group of \mathbb{R}_2^{n+3} keeping the Lorentzian product $\langle X, Y \rangle_2$ invariant. Then $O(n+3, 2)$ is a transformation group on \mathbb{Q}_1^{n+1} defined by

$$T([X]) = [XT], \quad X \in C^{n+2}, \quad T \in O(n+3, 2).$$

Topologically \mathbb{Q}_1^{n+1} is identified with the compact space $S^n \times S^1/S^0$, which is endowed by a standard Lorentzian metric $h = g_{S^n} \oplus (-g_{S^1})$, where g_{S^k} denotes the standard metric of the k -dimensional sphere S^k . Then \mathbb{Q}_1^{n+1} has conformal metric

$$[h] = \{e^\tau h | \tau \in C^\infty(\mathbb{Q}_1^{n+1})\}$$

and $[O(n+3, 2)]$ is the conformal transformation group of \mathbb{Q}_1^{n+1} (see [1, 18]).

Denoting $P = \{[X] \in \mathbb{Q}_1^{n+1} | x_1 = x_{n+2}\}$, $P_- = \{[X] \in \mathbb{Q}_1^{n+1} | x_{n+2} = 0\}$, $P_+ = \{[X] \in \mathbb{Q}_1^{n+1} | x_1 = 0\}$, we can define the following conformal diffeomorphisms,

$$\begin{aligned} \sigma_0 : \mathbb{R}_1^{n+1} &\rightarrow \mathbb{Q}_1^{n+1} \setminus P, & u &\mapsto [(\frac{\langle u, u \rangle_1 + 1}{2}, u, \frac{\langle u, u \rangle_1 - 1}{2})], \\ \sigma_1 : \mathbb{S}_1^{n+1}(1) &\rightarrow \mathbb{Q}_1^{n+1} \setminus P_+, & u &\mapsto [(1, u)], \\ \sigma_{-1} : \mathbb{H}_1^{n+1}(-1) &\rightarrow \mathbb{Q}_1^{n+1} \setminus P_-, & u &\mapsto [(u, 1)]. \end{aligned}$$

We may regard \mathbb{Q}_1^{n+1} as the common compactification of $\mathbb{R}_1^{n+1}, \mathbb{S}_1^{n+1}(1), \mathbb{H}_1^{n+1}(-1)$.

Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a spacelike hypersurface. Using σ_c , we obtain the hypersurface in \mathbb{Q}_1^{n+1} , $\sigma_c \circ x : M^n \rightarrow \mathbb{Q}_1^{n+1}$. From [1], we have the following theorem.

Theorem 2.1. *Two hypersurfaces $x, \bar{x} : M^n \rightarrow M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3, 2)$ such that $\sigma_c \circ x = T(\sigma_c \circ \bar{x}) : M^n \rightarrow \mathbb{Q}_1^{n+1}$.*

Since $x : M^n \rightarrow M_1^{n+1}(c)$ is a spacelike hypersurface, $(\sigma_c \circ x)_*(TM^n)$ is a positive definite subbundle of $T\mathbb{Q}_1^{n+1}$. For any local lift Z of the standard projection $\pi : C^{n+2} \rightarrow \mathbb{Q}_1^{n+1}$, we get a local lift $y = Z \circ \sigma_c \circ x : U \rightarrow C^{n+1}$ of $\sigma_c \circ x : M \rightarrow \mathbb{Q}_1^{n+1}$ in an open subset U of M^n . Thus $\langle dy, dy \rangle_2 = \rho^2 \langle dx, dx \rangle_s$ is a local metric, where $\rho \in C^\infty(U)$. We denote by Δ and κ the Laplacian operator and the normalized scalar curvature with

respect to the local positive definite metric $\langle dy, dy \rangle$, respectively. Similar to Wang's proof of Theorem 1.2 in [19], we can get the following theorem.

Theorem 2.2. *Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a spacelike hypersurface, then the 2-form $g = -(\langle \Delta y, \Delta y \rangle_2 - n^2 \kappa) \langle dy, dy \rangle_2$ is a globally defined conformal invariant. Moreover, g is positive definite at any non-umbilical point of M^n .*

We call g the conformal metric of the spacelike hypersurface M^n . There exists a unique lift

$$Y : M \rightarrow C^{n+2}$$

such that $g = \langle dY, dY \rangle_2$. We call Y the conformal position vector of the spacelike hypersurface M^n . Theorem 2.2 implies that

Theorem 2.3. *Two spacelike hypersurfaces $x, \bar{x} : M^n \rightarrow M_1^{n+1}(c)$ are conformally equivalent if and only if there exists $T \in O(n+3, 2)$ such that $\bar{Y} = YT$, where Y, \tilde{Y} are the conformal position vector of x, \tilde{x} , respectively.*

Let $\{E_1, \dots, E_n\}$ be a local orthonormal basis of M^n with respect to g with dual basis $\{\omega_1, \dots, \omega_n\}$. Denote $Y_i = E_i(Y)$ and define

$$N = -\frac{1}{n} \Delta Y - \frac{1}{2n^2} \langle \Delta Y, \Delta Y \rangle_2 Y,$$

where Δ is the Laplace operator of g , then we have

$$\langle N, Y \rangle_2 = 1, \quad \langle N, N \rangle_2 = 0, \quad \langle N, Y_k \rangle_2 = 0, \quad \langle Y_i, Y_j \rangle_2 = \delta_{ij}, \quad 1 \leq i, j, k \leq n.$$

We may decompose \mathbb{R}_2^{n+3} such that

$$\mathbb{R}_2^{n+3} = \text{span}\{Y, N\} \oplus \text{span}\{Y_1, \dots, Y_n\} \oplus \mathbb{V},$$

where $\mathbb{V} \perp \text{span}\{Y, N, Y_1, \dots, Y_n\}$. We call \mathbb{V} the conformal normal bundle of x , which is linear bundle. Let ξ be a local section of \mathbb{V} and $\langle \xi, \xi \rangle_2 = -1$, then $\{Y, N, Y_1, \dots, Y_n, \xi\}$

forms a moving frame in \mathbb{R}_2^{n+3} along M^n . We write the structure equations as follows,

$$\begin{aligned}
(2.1) \quad & dY = \sum_i \omega_i Y_i, \\
& dN = \sum_{ij} A_{ij} \omega_j Y_i + \sum_i C_i \omega_i \xi, \\
& dY_i = - \sum_j A_{ij} \omega_j Y - \omega_i N + \sum_j \omega_{ij} Y_j + \sum_j B_{ij} \omega_j \xi, \\
& d\xi = \sum_i C_i \omega_i Y + \sum_{ij} B_{ij} \omega_j Y_i,
\end{aligned}$$

where $\omega_{ij}(= -\omega_{ji})$ are the connection 1-forms on M^n with respect to $\{\omega_1, \dots, \omega_n\}$. It is clear that $A = \sum_{ij} A_{ij} \omega_j \otimes \omega_i$, $B = \sum_{ij} B_{ij} \omega_j \otimes \omega_i$, $C = \sum_i C_i \omega_i$ are globally defined conformal invariants. We call A , B and C the Blaschke tensor, the conformal second fundamental form and the conformal 1-form, respectively. The covariant derivatives of these tensors with respect to ω_{ij} are defined by:

$$\begin{aligned}
& \sum_j C_{i,j} \omega_j = dC_i + \sum_k C_k \omega_{kj}, \\
& \sum_k A_{ij,k} \omega_k = dA_{ij} + \sum_k A_{ik} \omega_{kj} + \sum_k A_{kj} \omega_{ki}, \\
& \sum_k B_{ij,k} \omega_k = dB_{ij} + \sum_k B_{ik} \omega_{kj} + \sum_k B_{kj} \omega_{ki}.
\end{aligned}$$

By exterior differentiation of structure equations (2.1), we can get the integrable conditions of the structure equations

$$\begin{aligned}
(2.2) \quad & A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \\
& A_{ij,k} - A_{ik,j} = B_{ij} C_k - B_{ik} C_j,
\end{aligned}$$

$$(2.3) \quad B_{ij,k} - B_{ik,j} = \delta_{ij} C_k - \delta_{ik} C_j,$$

$$(2.4) \quad C_{i,j} - C_{j,i} = \sum_k (B_{ik} A_{kj} - B_{jk} A_{ki}),$$

$$(2.5) \quad R_{ijkl} = B_{il} B_{jk} - B_{ik} B_{jl} + A_{ik} \delta_{jl} + A_{jl} \delta_{ik} - A_{il} \delta_{jk} - A_{jk} \delta_{il}.$$

Furthermore, we have

$$(2.6) \quad \begin{aligned} \operatorname{tr}(A) &= \frac{1}{2n}(n^2\kappa - 1), \quad R_{ij} = \operatorname{tr}(A)\delta_{ij} + (n-2)A_{ij} + \sum_k B_{ik}B_{kj}, \\ (1-n)C_i &= \sum_j B_{ij,j}, \quad \sum_{ij} B_{ij}^2 = \frac{n-1}{n}, \quad \sum_i B_{ii} = 0, \end{aligned}$$

where κ is the normalized scalar curvature of g . From (2.6), we see that when $n \geq 3$, all coefficients in the structure equations are determined by the conformal metric g and the conformal second fundamental form B , thus we get the following conformal congruent theorem.

Theorem 2.4. *Two spacelike hypersurfaces $x, \bar{x} : M^n \rightarrow M_1^{n+1}(c)$ ($n \geq 3$) are conformally equivalent if and only if there exists a diffeomorphism $\varphi : M^n \rightarrow M^n$ which preserves the conformal metric and the conformal second fundamental form.*

Next we give the relations between the conformal invariants and the isometric invariants of a spacelike hypersurface in $M_1^{n+1}(c)$.

First we consider the spacelike hypersurface $x : M^n \rightarrow \mathbb{R}_1^{n+1}$ in \mathbb{R}_1^{n+1} . Let $\{e_1, \dots, e_n\}$ be an orthonormal local basis with respect to the induced metric $I = \langle dx, dx \rangle_1$ with dual basis $\{\theta_1, \dots, \theta_n\}$. Let e_{n+1} be a normal vector field of x , $\langle e_{n+1}, e_{n+1} \rangle_1 = -1$. Let $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$ denote the second fundamental form, the mean curvature $H = \frac{1}{n} \sum_i h_{ii}$. Denote by Δ_M the Laplacian operator and κ_M the normalized scalar curvature for I . By structure equation of $x : M^n \rightarrow \mathbb{R}_1^{n+1}$ we get that

$$(2.7) \quad \Delta_M x = nHe_{n+1}.$$

There is a local lift of x

$$y : M^n \rightarrow C^{n+2}, \quad y = \left(\frac{\langle x, x \rangle_1 + 1}{2}, x, \frac{\langle x, x \rangle_1 - 1}{2} \right).$$

It follows from (2.7) that

$$\langle \Delta y, \Delta y \rangle_2 - n^2\kappa_M = \frac{n}{n-1}(-|II|^2 + n|H|^2) = -e^{2\tau}.$$

Therefore the conformal metric g , conformal position vector of x and ξ have the following expression,

$$(2.8) \quad \begin{aligned} g &= \frac{n}{n-1}(|II|^2 - n|H|^2) \langle dx, dx \rangle_1 := e^{2\tau}I, \quad Y = e^\tau y, \\ \xi &= -Hy + (\langle x, e_{n+1} \rangle_1, e_{n+1}, \langle x, e_{n+1} \rangle_1). \end{aligned}$$

By a direct calculation we get the following expression of the conformal invariants,

$$(2.9) \quad \begin{aligned} A_{ij} &= e^{-2\tau} [\tau_i \tau_j - h_{ij} H - \tau_{i,j} + \frac{1}{2} (-|\nabla \tau|^2 + |H|^2) \delta_{ij}], \\ B_{ij} &= e^{-\tau} (h_{ij} - H \delta_{ij}), \quad C_i = e^{-2\tau} (H \tau_i - H_i - \sum_j h_{ij} \tau_j), \end{aligned}$$

where $\tau_i = e_i(\tau)$ and $|\nabla \tau|^2 = \sum_i \tau_i^2$, and $\tau_{i,j}$ is the Hessian of τ for I and $H_i = e_i(H)$.

For a spacelike hypersurface $x : M^n \rightarrow \mathbb{S}_1^{n+1}(1)$, the conformal metric g , conformal position vector of x and ξ have the following expression,

$$(2.10) \quad \begin{aligned} g &= \frac{n}{n-1} (|II|^2 - n|H|^2) \langle dx, dx \rangle_1 := e^{2\tau} I, \\ Y &= e^\tau (1, x) = e^\tau y, \quad \xi = -Hy + (0, e_{n+1}). \end{aligned}$$

For a spacelike hypersurface $x : M^n \rightarrow \mathbb{H}_1^{n+1}(-1)$, the conformal metric g , conformal position vector of x and ξ have the following expression,

$$(2.11) \quad \begin{aligned} g &= \frac{n}{n-1} (|II|^2 - n|H|^2) \langle dx, dx \rangle_2 := e^{2\tau} I, \\ Y &= e^\tau (x, 1) = e^\tau y, \quad \xi = -Hy + (e_{n+1}, 0). \end{aligned}$$

Using the similar calculation from (2.10) and (2.11), we have the following united expression of the conformal invariants,

$$(2.12) \quad \begin{aligned} A_{ij} &= e^{-2\tau} [\tau_i \tau_j - \tau_{i,j} - h_{ij} H + \frac{1}{2} (-|\nabla \tau|^2 + |H|^2 + c) \delta_{ij}], \\ B_{ij} &= e^{-\tau} (h_{ij} - H \delta_{ij}), \quad C_i = e^{-2\tau} (H \tau_i - H_i - \sum_j h_{ij} \tau_j), \end{aligned}$$

where $c = 1$ for $x : M^n \rightarrow S_1^{n+1}(1)$, and $c = -1$ for $x : M^n \rightarrow H_1^{n+1}(-1)$.

3 Typical examples

In this section, we present some examples of the spacelike hypersurfaces in $M_1^{n+1}(c)$ with constant eigenvalues of para-Blaschke tensor.

Example 3.1. For constant $a > 0$, let $x_1 : \mathbb{H}^k(-1) \rightarrow \mathbb{R}_1^{k+1}$ be the standard embedding and $y : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{n-k}$ identity. We define the spacelike hypersurface

$$x = (x_1, y) : \mathbb{H}^k(-a) \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}_1^{n+1}, \quad 1 \leq k \leq n-1.$$

Let $\xi = (\frac{1}{a}x_1, \vec{0})$ be the normal vector field of x . Thus

$$I = \langle dx, dx \rangle_1 = I_{\mathbb{H}^k(-a)} + I_{\mathbb{R}^{n-k}}, \quad II = - \langle dx, d\xi \rangle_1 = \frac{-1}{a} I_{\mathbb{H}^k(-a)},$$

where $I_{\mathbb{H}^k(-a)}$ denotes the standard metric on $\mathbb{H}^k(-a)$ and $I_{\mathbb{R}^{n-k}}$ the standard metric on \mathbb{R}^{n-k} .

Let $\{e_1, \dots, e_k\}$ be a local fields of orthonormal basis on $\mathbb{H}^k(-a)$ and $\{e_{k+1}, \dots, e_n\}$ a local fields of orthonormal basis on \mathbb{R}^{n-k} , then $\{e_1, \dots, e_n\}$ is a local fields of orthonormal basis on $\mathbb{H}^k(-a) \times \mathbb{R}^{n-k}$. Thus, under the local fields of orthonormal basis $\{e_1, \dots, e_n\}$,

$$(h_{ij}) = \text{diag}\left(\frac{-1}{a}, \dots, \frac{-1}{a}, 0, \dots, 0\right).$$

Under the local fields of orthonormal basis, from (2.9), we have

$$(B_{ij}) = \text{diag}(\underbrace{b_1, \dots, b_1}_k, \underbrace{b_2, \dots, b_2}_{n-k}), \quad (A_{ij}) = \text{diag}(\underbrace{a_1, \dots, a_1}_k, \underbrace{a_2, \dots, a_2}_{n-k}),$$

where

$$b_1 = \frac{1}{n} \sqrt{\frac{(n-1)(n-k)}{k}}, \quad b_2 = \frac{-1}{n} \sqrt{\frac{(n-1)k}{n-k}}, \quad a_1 = \frac{(n-1)(k-2n)}{2n^2(n-k)}, \quad a_2 = \frac{(n-1)k}{2n^2(n-k)}.$$

Thus $(D_{ij}^\lambda) = \text{diag}(\underbrace{d_1, \dots, d_1}_k, \underbrace{d_2, \dots, d_2}_{n-k})$ and $d_1 = a_1 + \lambda b_1$, $d_2 = a_2 + \lambda b_2$.

Example 3.2. Let $x_1 : \mathbb{S}^k(1) \rightarrow \mathbb{R}^{k+1}$ and $x_2 : \mathbb{H}^{n-k}(-1) \rightarrow \mathbb{R}_1^{n-k+1}$ be two standard embeddings. For constant $a > 0$, we define the spacelike hypersurface

$$x = (\sqrt{1+a^2}x_1, ax_2) : \mathbb{S}^k(\sqrt{1+a^2}) \times \mathbb{H}^{n-k}(-a) \rightarrow \mathbb{S}_1^{n+1}(1) \subset \mathbb{R}_1^{n+2}, \quad 1 \leq k \leq n-1.$$

Let $\xi = (ax_1, \sqrt{1+a^2}x_2)$ be the normal vector field of x . Thus

$$I = \langle dx, dx \rangle_1 = (1+a^2)I_{\mathbb{S}^k(1)} + a^2I_{\mathbb{H}^{n-k}(-1)},$$

$$II = - \langle dx, d\xi \rangle_1 = -a\sqrt{1+a^2}(I_{\mathbb{S}^k(1)} + I_{\mathbb{H}^{n-k}(-1)}).$$

Let $\{e_1, \dots, e_k\}$ be a local fields of orthonormal basis on $\mathbb{S}^k(\sqrt{1+a^2})$ and $\{e_{k+1}, \dots, e_n\}$ a local fields of orthonormal basis on $\mathbb{H}^{n-k}(-a)$, then $\{e_1, \dots, e_n\}$ is a local fields of

orthonormal basis on $\mathbb{S}^k(\sqrt{1+a^2}) \times \mathbb{H}^{n-k}(-a)$. Thus, under the local fields of orthonormal tangent frame $\{e_1, \dots, e_n\}$,

$$(h_{ij}) = \text{diag}\left(\frac{-a}{\sqrt{1+a^2}}, \dots, \frac{-a}{\sqrt{1+a^2}}, \frac{-\sqrt{1+a^2}}{a}, \dots, \frac{-\sqrt{1+a^2}}{a}\right).$$

Under the local fields of orthonormal basis, from (2.11), we have

$$(B_{ij}) = \text{diag}(\underbrace{b_1, \dots, b_1}_k, \underbrace{b_2, \dots, b_2}_{n-k}), \quad (A_{ij}) = \text{diag}(\underbrace{a_1, \dots, a_1}_k, \underbrace{a_2, \dots, a_2}_{n-k}),$$

where

$$b_1 = \frac{1}{n} \sqrt{\frac{(n-1)(n-k)}{k}}, \quad b_2 = \frac{-1}{n} \sqrt{\frac{(n-1)k}{n-k}},$$

$$a_1 = \frac{n-1}{k(n-k)} \frac{(n-k)^2 + n^2 a^2}{2n^2}, \quad a_2 = \frac{n-1}{k(n-k)} \frac{k^2 - n^2 a^2 - n^2}{2n^2}.$$

Thus $(D_{ij}^\lambda) = \text{diag}(\underbrace{d_1, \dots, d_1}_k, \underbrace{d_2, \dots, d_2}_{n-k})$ and $d_1 = a_1 + \lambda b_1$, $d_2 = a_2 + \lambda b_2$.

Example 3.3. Let $x_1 : \mathbb{H}^k(-1) \rightarrow \mathbb{R}_1^{k+1}$ and $x_2 : \mathbb{H}^{n-k}(-1) \rightarrow \mathbb{R}_1^{n-k+1}$ be two standard embeddings. For constant a satisfying $0 < a < 1$, We we define the spacelike hypersurface

$$x = (\sqrt{1-a^2}x_1, ax_2) : \mathbb{H}^k(-a) \times \mathbb{H}^{n-k}(-\sqrt{1-a^2}) \rightarrow \mathbb{H}_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2}, \quad 1 \leq k \leq n-1.$$

Let $\xi = (-ax_1, \sqrt{1-a^2}x_2)$ be the normal vector field of x . Thus

$$I = \langle dx, dx \rangle_1 = (1-a^2)I_{H^k(-1)} + a^2 I_{H^{n-k}(-1)},$$

$$II = - \langle dx, d\xi \rangle_1 = a\sqrt{1-a^2}(I_{H^k(-1)} - I_{H^{n-k}(-1)}).$$

Let $\{e_1, \dots, e_k\}$ be a local fields of orthonormal basis on $\mathbb{H}^k(-a)$ and $\{e_{k+1}, \dots, e_n\}$ a local fields of orthonormal basis on $\mathbb{H}^{n-k}(-\sqrt{1-a^2})$, then $\{e_1, \dots, e_n\}$ is a local fields of orthonormal basis on $\mathbb{H}^k(-a) \times \mathbb{H}^{n-k}(-\sqrt{1-a^2})$. Thus, under the local fields of orthonormal basis $\{e_1, \dots, e_n\}$,

$$(h_{ij}) = \text{diag}\left(\frac{a}{\sqrt{1-a^2}}, \dots, \frac{a}{\sqrt{1-a^2}}, \frac{-\sqrt{1-a^2}}{a}, \dots, \frac{-\sqrt{1-a^2}}{a}\right).$$

Under the local fields of orthonormal basis, from (2.11), we have

$$(B_{ij}) = \text{diag}(\underbrace{b_1, \dots, b_1}_k, \underbrace{b_2, \dots, b_2}_{n-k}), \quad (A_{ij}) = \text{diag}(\underbrace{a_1, \dots, a_1}_k, \underbrace{a_2, \dots, a_2}_{n-k}),$$

where

$$b_1 = \frac{1}{n} \sqrt{\frac{(n-1)(n-k)}{k}}, \quad b_2 = \frac{-1}{n} \sqrt{\frac{(n-1)k}{n-k}},$$

$$a_1 = \frac{n-1}{k(n-k)} \frac{(n-k)^2 - n^2 a^2}{2n^2}, \quad a_2 = \frac{n-1}{k(n-k)} \frac{n^2 a^2 - n^2 + k^2}{2n^2}.$$

Thus $(D_{ij}^\lambda) = \text{diag}(\underbrace{d_1, \dots, d_1}_k, \underbrace{d_2, \dots, d_2}_{n-k})$ and $d_1 = a_1 + \lambda b_1$, $d_2 = a_2 + \lambda b_2$.

Example 3.4. Let p, q be any two given natural numbers with $p + q < n$ and a real number $a > 1$. We define the spacelike hypersurface

$$x : \mathbb{H}^q(-\sqrt{a^2 - 1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \rightarrow \mathbb{R}_1^{n+1},$$

defined by

$$x(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2 - 1})$, $u'' \in \mathbb{S}^p(a)$, $u''' \in \mathbb{R}^{n-p-q-1}$.

Let $b = \sqrt{a^2 - 1}$. One of the normal vector of x can be taken as

$$e_{n+1} = \left(\frac{a}{b}u', \frac{b}{a}u'', 0\right).$$

The first and second fundamental form of x are given by

$$I = t^2(< du', du' >_1 + du'' \cdot du'') + dt \cdot dt + du''' \cdot du''',$$

$$II = - < dx, de_{n+1} >_1 = -t\left(\frac{a}{b} < du', du' >_1 + \frac{b}{a} du'' \cdot du''\right).$$

Thus the mean curvature of x satisfies

$$H = \frac{-pb^2 - qa^2}{nabt},$$

$$\text{and } e^{2\tau} = \frac{n}{n-1} [\sum_{ij} h_{ij}^2 - nH^2] = \frac{p(n-p)b^4 - 2pqa^2b^2 + q(n-q)a^4}{(n-1)t^2} := \frac{\alpha^2}{t^2}.$$

From (2.8) and (2.12), we see that the conformal 1-form $C = 0$, and the conformal metric and the conformal second fundamental form of x are given by

$$g = \alpha^2 < du', du' > + \alpha^2 du'' \cdot du'' + \frac{\alpha^2}{t^2} (dt \cdot dt + du''' \cdot du''') = \tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3,$$

$$(3.13) \quad B = \sum_{ij} B_{ij} \omega_i \otimes \omega_j, \quad (B_{ij}) = (\underbrace{b_1, \dots, b_1}_q, \underbrace{b_2, \dots, b_2}_p, \underbrace{b_3, \dots, b_3}_{n-p-q}),$$

$$A = \sum_{ij} A_{ij} \omega_i \otimes \omega_j, \quad (A_{ij}) = (\underbrace{a_1, \dots, a_1}_q, \underbrace{a_2, \dots, a_2}_p, \underbrace{a_3, \dots, a_3}_{n-p-q}),$$

where $b_1 = \frac{pb^2 - (n-q)a^2}{nab\alpha}$, $b_2 = \frac{qa^2 - (n-p)b^2}{nab\alpha}$, $b_3 = \frac{pb^2 + qa^2}{nab\alpha}$, and

$$a_1 = \frac{(pb^2 + qa^2)^2 - (pb^2 + qa^2)2na^2 + n^2a^2b^2}{2n^2a^2b^2\alpha^2},$$

$$a_2 = \frac{(pb^2 + qa^2)^2 - (pb^2 + qa^2)2nb^2 + n^2a^2b^2}{2n^2a^2b^2\alpha^2}, a_3 = \frac{(pb^2 + qa^2)^2 + n^2a^2b^2}{2n^2a^2b^2\alpha^2}.$$

Thus $(D_{ij}^\lambda) = \text{diag}(\underbrace{d_1, \dots, d_1}_q, \underbrace{d_2, \dots, d_2}_p, \underbrace{d_3, \dots, d_3}_{n-p-q})$, $d_i = a_i + \lambda b_i$, $i = 1, 2, 3$.

Example 3.5. Given constants $\lambda, r(r > 0)$, we define the spacelike hypersurface

$$x = \left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right) : M^k \times \mathbb{H}^{n-k}(-r) \rightarrow \mathbb{S}_1^{n+1}(1), \quad 2 \leq k \leq n-1.$$

Here $y = (y_0, y_2) : \mathbb{H}^{n-k}(-r) \rightarrow \mathbb{R}_1^{n-k+1}$ is a standard embedding, and $y_1 : M^k \rightarrow \mathbb{S}_1^{k+1}(r) \subset \mathbb{R}_1^{k+2}$ is a umbilic-free spacelike hypersurface with constant scalar curvature R_1 and the mean curvature H_1 satisfying $R_1 = \frac{nk(k-1) + (n-1)r^2}{nr^2} - n(n-1)\lambda^2$, $H_1 = \frac{n}{k}\lambda$, respectively.

Using the structure of the spacelike hypersurface y_1 , we have

$$\Delta_1 y_1 = -\frac{ny_1}{r^2} + n\lambda\xi_1,$$

where Δ_1 is the Laplacian with respect to the first fundamental form $\langle dy_1, dy_1 \rangle_1$ and ξ_1 is the unit normal vector field of y_1 .

The standard embedding $y = (y_0, y_2) : \mathbb{H}^{n-k}(-r) \rightarrow \mathbb{R}_1^{n-k+1}$ is totally umbilical, thus the scalar curvature $R_2 = -\frac{(n-k)(n-k-1)}{r^2}$, and

$$\Delta_2 y = \frac{(n-k)y}{r^2},$$

where Δ_2 is the Laplacian with respect to the first fundamental form $\langle dy, dy \rangle_1$.

The conformal position vector of the spacelike hypersurface is

$$Y = (y_0, y_1, y_2) : M^k \times \mathbb{H}^{n-k}(-r) \rightarrow \mathbb{R}_2^{n+3}.$$

Since the conformal metric $g = \langle dY, dY \rangle$,

$$N = \left(-\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)y_0, \left(\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)y_1 - \lambda\xi_1, \left(-\frac{1}{2r^2} + \frac{\lambda^2}{2}\right)y_2.$$

We take a local orthonormal basis $\{e_p, p = 1, \dots, k\}$ on TM^k , and $\{e_q, q = k + 1, \dots, n\}$ on $T\mathbb{H}^{n-k}(-r)$. Thus $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ is a local orthonormal basis on $T(M^k \times \mathbb{H}^{n-k}(-r))$ and

$$Y_i = (0, e_i(y_1), \vec{0}), \quad 1 \leq i \leq k, \quad Y_j = (e_j(y_0), 0, e_j(y_2)), \quad k + 1 \leq j \leq n, \quad \xi = (0, \xi_1, \vec{0}).$$

Under the local basis $\{e_1, \dots, e_n\}$, using $A_{ij} = \langle Y_i, N_j \rangle_2, B_{ij} = \langle Y_i, \xi_j \rangle_2$, and $D_{ij} = A_{ij} + \lambda B_{ij}$, we have $C = 0$ and

$$(3.14) \quad \begin{aligned} (A_{ij}) &= \left(\frac{1 + \lambda^2 r^2}{2r^2} \delta_{ij} - \lambda h_{ij} \right) \oplus \left(\frac{\lambda^2 r^2 - 1}{2r^2} \delta_{st} \right), \quad 1 \leq i, j \leq k, \quad k + 1 \leq s, t \leq n, \\ (B_{ij}) &= (h_{ij} - \lambda \delta_{ij}) \oplus (-\lambda \delta_{st}), \quad 1 \leq i, j \leq k, \quad k + 1 \leq s, t \leq n, \\ (D_{ij}^\lambda) &= \frac{1 - \lambda^2 r^2}{2r^2} I_m \oplus \left(-\frac{1 + \lambda^2 r^2}{2r^2} \right) \delta_{st}, \quad 1 \leq i, j \leq k, \quad k + 1 \leq s, t \leq n. \end{aligned}$$

Example 3.6. Given constants $\lambda, r (r > 0)$, let

$$y = (y_0, \tilde{y}_0, y_1) : M^k \rightarrow \mathbb{H}_1^{k+1}(-r) \subset \mathbb{R}_2^{k+2}, \quad 2 \leq k \leq n - 1,$$

be a spacelike hypersurface with constant scalar curvature R_1 and the mean curvature H_1 satisfying $R_1 = \frac{-nk(k-1)+(n-1)r^2}{nr^2} - n(n-1)\lambda^2$, $H_1 = \frac{n}{k}\lambda$, respectively.

Since $-y_0^2 - \tilde{y}_0^2 + \langle y_1, y_1 \rangle = -r^2$, y_0 and \tilde{y}_0 can not be zero simultaneously. Without loss of generality, we assume that $y_0 \neq 0$. In this case, we define the spacelike hypersurface

$$x = \left(\frac{\tilde{y}_0}{|y_0|}, \frac{y_1}{|y_0|}, \frac{y_2}{|y_0|} \right) : M^k \times \mathbb{S}^{n-k}(r) \rightarrow \mathbb{S}_1^{n+1}(1),$$

where $y_2 : \mathbb{S}^{n-k}(r) \rightarrow \mathbb{R}^{n-k+1}$ is a round sphere with radius r .

Using the structure equation of the spacelike hypersurface y , we have

$$\Delta_1 y = \frac{ky}{r^2} + n\lambda \xi_1,$$

where Δ_1 is the Laplacian with respect to the first fundamental form $\langle dy, dy \rangle_1$ and ξ_1 is the unit normal vector field of y .

The round sphere $y_2 : \mathbb{S}^{n-k}(r) \rightarrow \mathbb{R}^{n-k+1}$ is totally umbilical, thus the scalar curvature is $R_2 = \frac{(n-k)(n-k-1)}{r^2}$, and

$$\Delta_2 y_2 = -\frac{(n-k)y_2}{r^2},$$

where \triangle_2 is the Laplacian with respect to the first fundamental form $\langle dy_2, dy_2 \rangle$.

The conformal position vector of the spacelike hypersurface is

$$Y = (y, y_2) : M^k \times \mathbb{S}^{n-k}(r) \rightarrow \mathbb{R}_2^{n+3}.$$

Since the conformal metric $g = \langle dY, dY \rangle$, we have

$$N = ((-\frac{1}{2r^2} + \frac{\lambda^2}{2})y - \lambda\xi_1, (\frac{1}{2r^2} + \frac{\lambda^2}{2})y_2).$$

We take a local orthonormal basis $\{e_p, p = 1, \dots, k\}$ on TM^k , and $\{e_q, q = k+1, \dots, n\}$ on $T\mathbb{S}^{n-k}(r)$. Thus $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ is a local orthonormal basis on $T(M^k \times \mathbb{S}^{n-k}(r))$ and

$$Y_i = (e_i(y), \vec{0}), \quad 1 \leq i \leq k, \quad Y_j = (\vec{0}, e_j(y_2)), \quad k+1 \leq j \leq n, \quad \xi = (\xi_1, \vec{0}).$$

Under the local basis $\{e_1, \dots, e_n\}$, using $A_{ij} = \langle Y_i, N_j \rangle$, $B_{ij} = \langle Y_i, \xi_j \rangle$, we have $C = 0$ and

$$\begin{aligned} (A^{ij}) &= (\frac{\lambda^2 r^2 - 1}{2r^2} \delta_{ij} - \lambda h_{ij}) \oplus (\frac{\lambda^2 r^2 + 1}{2r^2} \delta_{st}), \quad 1 \leq i, j \leq k, \quad k+1 \leq s, t \leq n, \\ (3.15) \quad (B_{ij}) &= (h_{ij} - \lambda \delta_{ij}) \oplus (-\lambda \delta_{st}), \quad 1 \leq i, j \leq k, \quad k+1 \leq s, t \leq n, \\ (D_{ij}^\lambda) &= -\frac{1 + \lambda^2 r^2}{2r^2} \delta_{ij} \oplus \frac{1 - \lambda^2 r^2}{2r^2} \delta_{st}, \quad 1 \leq i, j \leq k, \quad k+1 \leq s, t \leq n. \end{aligned}$$

In [8], authors classified completely the conformal isoparametric spacelike hypersurfaces in $M_1^{n+1}(c)$.

Theorem 3.1. [8] *Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a spacelike hypersurface in $M_1^{n+1}(c)$ with two distinct principal curvatures. If the conformal form vanishes, then locally x is conformally equivalent to one of the following hypersurfaces*

- (1), $\mathbb{S}^k(\sqrt{a^2 + 1}) \times \mathbb{H}^{n-k}(-a) \subset \mathbb{S}_1^{n+1}(1)$, $a > 0$, $1 \leq k \leq n-1$;
- (2), $\mathbb{H}^k(-a) \times \mathbb{H}^{n-k}(-\sqrt{1-a^2}) \subset \mathbb{H}_1^{n+1}(-1)$, $0 < a < 1$, $1 \leq k \leq n-1$;
- (3), $\mathbb{H}^k(-a) \times \mathbb{R}^{n-k} \subset \mathbb{R}_1^{n+1}$, $a > 0$, $1 \leq k \leq n-1$.

Theorem 3.2. [8] *Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a conformal isoparametric spacelike hypersurface in $M_1^{n+1}(c)$ with r distinct principal curvatures. If $r \geq 3$, then $r = 3$, and locally x is conformally equivalent to the following hypersurface*

$$x : \mathbb{H}^q(-\sqrt{a^2 - 1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \rightarrow \mathbb{R}_1^{n+1},$$

defined by

$$x(u', u'', t, u''') = (tu', tu'', u'''),$$

where $u' \in \mathbb{H}^q(-\sqrt{a^2 - 1})$, $u'' \in \mathbb{S}^p(a)$, $u''' \in \mathbb{R}^{n-p-q-1}$, $a > 1$.

The following theorem is need in the proof of the main theorem, readers refer [7].

Theorem 3.3. [7] *Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a spacelike hypersurface without umbilical points. If conformal invariants of x satisfy*

$$(1), C = 0, \quad (2), A = \mu B + \lambda g,$$

Then x is conformally equivalent to a spacelike hypersurface with constant mean curvature and constant scalar curvature.

4 Proof of the main Theorem

Proof of Theorem 1.1. From Theorem 3.1, Theorem 3.2, Example 3.1, Example 3.2, Example 3.3 and Example 3.4, we know that if the spacelike hypersurface is conformally isoparametric, then the spacelike hypersurface is also para-Blaschke isoparametric.

Next we assume that the spacelike hypersurface is a para-Blaschke isoparametric spacelike hypersurface and the number of the distinct eigenvalues of the para-Blaschke tensor D^λ is more than two. Since the conformal 1-form vanishes, we can have a local orthonormal basis $\{E_1, \dots, E_n\}$ such that

$$(B_{ij}) = \text{diag}(b_1, \dots, b_n), \quad (A_{ij}) = \text{diag}(a_1, \dots, a_n), \quad (D_{ij}^\lambda) = \text{diag}(d_1, d_2, \dots, d_n).$$

Using the covariant derivative $dD_{ij}^\lambda + \sum_k D_{kj}^\lambda \omega_{ki} + \sum_k D_{ik}^\lambda \omega_{kj} = \sum_k D_{ij,k}^\lambda \omega_k$, we have

$$(4.16) \quad (d_i - d_j)\omega_{ij} = \sum_k D_{ij,k}^\lambda \omega_k.$$

For each i fixed, we define the index set $[i] = \{m | d_m = d_i\}$. We have the following results

$$(4.17) \quad \begin{aligned} D_{ij,k}^\lambda &= 0, \quad \text{when } [i] = [j], \text{ or } [i] = [k], \text{ or } [j] = [k]. \\ \omega_{ij} &= \sum_{k \notin [i], [j]} \frac{D_{ij,k}^\lambda}{d_i - d_j} \omega_k, \quad \text{when } [i] \neq [j]. \end{aligned}$$

The second covariant derivative of D_{ij}^λ is defined by

$$\sum_l D_{ij,k}^\lambda \omega_l = dD_{ij,k}^\lambda + \sum_l (D_{lj,k}^\lambda \omega_{li} + D_{il,k}^\lambda \omega_{lj} + D_{ij,l}^\lambda \omega_{lk}).$$

Let $[i] \neq [j]$, we have

$$D_{ij,ij}^\lambda = \sum_{k \notin [i],[j]} \frac{2(D_{ij,k}^\lambda)^2}{d_k - d_i}, \quad D_{ij,ji}^\lambda = \sum_{k \notin [i],[j]} \frac{2(D_{ij,k}^\lambda)^2}{d_k - d_j}.$$

Using the Ricci identities $D_{ij,ij}^\lambda - D_{ij,ji}^\lambda = \sum_m (D_{mj}^\lambda R_{mij} + D_{im}^\lambda R_{mji})$, we get

$$(4.18) \quad R_{ijij} = \sum_{k \notin [i],[j]} \frac{2(D_{ij,k}^\lambda)^2}{(d_k - d_i)(d_k - d_j)}.$$

For the conformal second fundamental form B , we have

$$(b_i - b_j)\omega_{ij} = \sum_k B_{ij,k}\omega_k.$$

Using (4.17), we get

$$(4.19) \quad (b_i - b_j) \frac{D_{ij,k}^\lambda}{d_i - d_j} = B_{ij,k}, \quad [i] \neq [j].$$

Let $k = i$ in (4.19), and using $D_{ij,i}^\lambda = 0$ we obtain

$$(4.20) \quad E_j(b_i) = B_{ii,j} = B_{ij,i} = 0, \quad [i] \neq [j].$$

In order to prove that b_i is a constant, we only need to prove

$$E_j(b_i) = 0, \quad [i] = [j].$$

For each i fixed, we consider two cases:

Case 1. There exist j, k such that

$$D_{ij,k}^\lambda \neq 0, \quad [j] \neq [i], \quad [k] \neq [i], \quad [j] \neq [k].$$

Case 2. For all j, k , we have $D_{ij,k}^\lambda = 0$.

Now we consider Case 1, since

$$D_{ij,k}^\lambda \neq 0, \quad [j] \neq [i], \quad [k] \neq [i], \quad [j] \neq [k],$$

from (4.19), we get

$$\frac{b_i - b_j}{d_i - d_j} = \frac{B_{ij,k}}{D_{ij,k}^\lambda} = \frac{B_{jk,i}}{D_{jk,i}^\lambda} = \frac{b_k - b_j}{d_k - d_j}.$$

Thus

$$b_i = (b_k - b_j) \frac{d_i - d_j}{d_k - d_j} + b_j.$$

From (4.20), since $E_l(b_k) = E_l(b_j) = 0$, $l \in [i]$, we have

$$(4.21) \quad E_l(b_i) = 0, \quad [i] = [l].$$

For Case 2. Since $D_{ij,k}^\lambda = 0$, $\forall j, k$, from (4.18), we get

$$R_{ijij} = 0, \quad j \notin [i].$$

From (2.5) we have $0 = R_{ijij} = -b_i b_j + a_i + a_j$, $j \notin [i]$. Since $a_i = d_i - \lambda b_i$, we have

$$-b_i b_j + d_i + d_j = \lambda(b_i + b_j), \quad j \notin [i].$$

Note that the number of the distinct eigenvalues of the para-Blaschke tensor D^λ is more than two, we can take j, k such that $[j] \neq [i]$, $[k] \neq [i]$ and $[k] \neq [j]$, and

$$-b_i b_k + d_i + d_k = \lambda(b_i + b_k), \quad j \notin [i].$$

Thus

$$-(b_i + \lambda)(b_j - b_k) = d_k - d_j.$$

namely

$$b_i = \frac{d_j - d_k}{b_j - b_k} + \lambda.$$

Noting $E_l(b_j) = E_l(b_k) = 0$, $l \in [i]$, $j, k \notin [i]$, we obtain

$$(4.22) \quad E_l(b_i) = 0, \quad l \in [i].$$

From (4.20), (4.21) and (4.22), it concludes that

$$E_j(b_i) = 0, \quad 1 \leq i, j \leq n.$$

Thus $\{b_i | i = 1, \dots, n\}$ are constant and x is conformal isoparametric spacelike hypersurface. Thus we complete the proof of Theorem 1.1.

Next we divide Theorem 1.2 into three cases. If the number of the distinct eigenvalues of the para-Blaschke tensor is 1, using Theorem 3.3, then we have the following proposition.

Proposition 4.1. *Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a para-Blaschke isoparametric spacelike hypersurface with r distinct eigenvalues of the para-Blaschke tensor. If $r = 1$, then x is conformally equivalent to a spacelike hypersurface with constant mean curvature and constant scalar curvature in $M_1^{n+1}(c)$.*

If the number of the distinct eigenvalues of the para-Blaschke tensor is more than two, then we have the following Proposition by Theorem 3.2 and Theorem 1.1.

Proposition 4.2. *Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a para-Blaschke isoparametric spacelike hypersurface with r distinct eigenvalues of the para-Blaschke tensor. If $r \geq 3$, then $r = 3$, and locally x is conformally equivalent to the following hypersurface,*

$$x : \mathbb{H}^q(-\sqrt{a^2 - 1}) \times \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \rightarrow \mathbb{R}_1^{n+1},$$

defined by $x(u', u'', t, u''') = (tu', tu'', u''')$, where $u' \in \mathbb{H}^q(-\sqrt{a^2 - 1})$, $u'' \in \mathbb{S}^p(a)$, $u''' \in \mathbb{R}^{n-p-q-1}$, $a > 1$.

Next we assume that the number of the distinct eigenvalues of the para-Blaschke tensor is two, we have

Proposition 4.3. *Let $x : M^n \rightarrow M_1^{n+1}(c)$ be a para-Blaschke isoparametric spacelike hypersurface with two distinct eigenvalues of the para-Blaschke tensor. Then x is locally conformal equivalent to one of the following hypersurfaces:*

- (1), $\mathbb{S}^k(\sqrt{a^2 + 1}) \times \mathbb{H}^{n-k}(-a) \subset \mathbb{S}_1^{n+1}(1)$, $a > 0$, $1 \leq k \leq n - 1$;
- (2), $\mathbb{H}^k(-a) \times \mathbb{H}^{n-k}(-\sqrt{1 - a^2}) \subset \mathbb{H}_1^{n+1}(-1)$, $0 < a < 1$, $1 \leq k \leq n - 1$;
- (3), $\mathbb{H}^k(-a) \times \mathbb{R}^{n-k} \subset \mathbb{R}_1^{n+1}$, $a > 0$, $0 \leq k \leq n - 1$;
- (4) the spacelike hypersurfaces defined by Example 3.5;
- (5) the spacelike hypersurfaces defined by Example 3.6.

Proof. Since the conformal 1-form vanishes, we can get a local orthonormal basis $\{E_1, \dots, E_n\}$ such that

$$(B_{ij}) = \text{diag}(b_1, \dots, b_n), \quad (A_{ij}) = \text{diag}(a_1, \dots, a_n), \quad (D_{ij}^\lambda) = \text{diag}(d_1, d_2, \dots, d_n).$$

Furthermore, we assume that

$$d_1 = d_2 = \dots = d_{m_1} = \mu, \quad d_{m_1+1} = d_{m_1+2} = \dots = d_n = \nu, \quad \mu \neq \nu.$$

Making the following convention on the ranges of indices:

$$1 \leq p, q, r, \dots \leq m_1, \quad m_1 + 1 \leq \alpha, \beta, \gamma, \dots \leq n, \quad 1 \leq i, j, k \leq n.$$

From (4.16), for all $i, j, p, q, \alpha, \beta$ we have

$$(4.23) \quad D_{ij,i}^\lambda = D_{ii,j}^\lambda = 0, \quad D_{pq,i}^\lambda = D_{\alpha\beta,i}^\lambda = 0, \quad \omega_{p\alpha} = \frac{\sum D_{p\alpha,k}^\lambda \omega_k}{\mu_1 - \mu_2}.$$

Since μ and ν are constant, using the total symmetry of $D_{ij,k}$, we can get that D^λ is parallel, i.e., $D_{ij,k} = 0, \forall i, j, k$.

Let V_1 and V_2 be the eigen-subbundles of the tangent bundle TM corresponding to μ, ν , respectively. Then

$$TM^n = V_1 \oplus V_2.$$

Since D^λ is parallel, we have

$$(4.24) \quad \omega_{p\alpha} = 0, \quad 1 \leq p \leq m_1, \quad m_1 + 1 \leq \alpha \leq n,$$

which implies that the Riemannian manifold (M^n, g) can be decomposed locally into a direct product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) , that is

$$(M, g) = (M_1, g_1) \times (M_2, g_2).$$

Thus $R_{p\alpha p\alpha} = 0$, and from (2.5) we know that

$$(4.25) \quad -b_p b_\alpha + a_p + a_\alpha = 0, \quad 1 \leq p \leq m_1, \quad m_1 + 1 \leq \alpha \leq n.$$

Claim 1: *The eigenvalues of the conformal second fundamental form B satisfy either $b_1 = b_2 = \dots = b_{m_1}$, or $b_{m_1+1} = \dots = b_n$.*

Proof of Claim 1: We assume that $n \geq 3$. From (4.25), we have

$$-b_p b_\alpha - \lambda b_p - \lambda b_\alpha + \mu + \nu = 0, \quad \forall p, \alpha,$$

that is

$$(4.26) \quad -(b_p + \lambda)(b_\alpha + \lambda) + \lambda^2 + \mu + \nu = 0, \quad \forall p, \alpha.$$

If $\lambda^2 + \mu + \nu \neq 0$, (4.26) implies $b_p + \lambda \neq 0$, $b_\alpha + \lambda \neq 0$, $\forall p, \alpha$. Since $m_1 \geq 2$, we have $-(b_q + \lambda)(b_\alpha + \lambda) + \lambda^2 + \mu + \nu = 0$, which implies that

$$(b_p - b_q)(b_\alpha + \lambda) = 0. \quad \forall p \neq q, \alpha.$$

Thus

$$b_1 = b_2 = \dots = b_{m_1}.$$

Similarly, if $n - m_1 \geq 2$, we can obtain $b_{m_1+1} = b_{m_1+2} = \dots = b_n$. Thus the conformal second fundamental form (B_{ij}) has two distinct constant eigenvalues.

If $\lambda^2 + \mu + \nu = 0$, (4.26) implies $(b_p + \lambda)(b_\alpha + \lambda) = 0$, which proves the Claim 1.

By Claim 1, the proof of the Proposition 4.3 is divided into the following two cases.

Case I. The conformal second fundamental form B has only two distinct eigenvalues. According to Theorem 3.1, x is locally conformal equivalent to one of the hypersurfaces in Example 3.1, Example 3.2 and Example 3.3.

Case II. The conformal second fundamental form B has more than two distinct eigenvalues.

By Claim 1, we see that either $b_1 = b_2 = \dots = b_{m_1}$, or $b_{m_1+1} = b_{m_1+2} = \dots = b_n$. Without loss of generality, assuming $b_{m_1+1} = b_{m_1+2} = \dots = b_n$. From the proof of the Claim 1, we know that $b_{m_1+1} = b_{m_1+2} = \dots = b_n = -\lambda$, $m_1 \geq 2$ and $\lambda^2 + \mu + \nu = 0$.

Let $\tilde{h}_{pq} = B_{pq} + \lambda\delta_{pq}$, then from (2.5), we obtain the components of curvature tensor on (M_1, g_1)

$$\begin{aligned} R_{pqst} &= -B_{ps}B_{qt} + B_{pt}B_{qs} + (\mu\delta_{ps} - \lambda B_{ps})\delta_{qt} + (\mu\delta_{qt} - \lambda B_{qt})\delta_{ps} \\ (4.27) \quad &- (\mu\delta_{pt} - \lambda B_{pt})\delta_{qs} - (\mu\delta_{qs} - \lambda B_{qs})\delta_{pt} \\ &= (2\mu + \lambda^2)(\delta_{ps}\delta_{qt} - \delta_{pt}\delta_{qs}) + \tilde{h}_{pt}\tilde{h}_{qs} - \tilde{h}_{ps}\tilde{h}_{qt}. \end{aligned}$$

The components of curvature tensor on (M_2, g_2)

$$(4.28) \quad R_{\alpha\beta\gamma\eta} = (2\nu + \lambda^2)(\delta_{\alpha\gamma}\delta_{\beta\eta} - \delta_{\beta\gamma}\delta_{\alpha\eta}).$$

Hence if $n - m_1 \geq 2$, (M_2, g_2) is of constant sectional curvature $2\nu + \lambda^2$.

Since $(2\mu + \lambda^2) + (2\nu + \lambda^2) = 2(\lambda^2 + \mu + \nu) = 0$ and $\mu \neq \nu$, we need to consider the following two subcases:

Subcase 2.1. $2\mu + \lambda^2 > 0$, $2\nu + \lambda^2 < 0$.

Set $r = (2\mu + \lambda^2)^{-\frac{1}{2}}$, then $2\nu + \lambda^2 = -r^{-2}$, and (M_2, g_2) can be locally identified with $\mathbb{H}^{n-m_1}(-r)$. Let $v : \mathbb{H}^{n-m_1}(-r) \rightarrow \mathbb{R}_1^{n-m_1+1}$ be the standard totally umbilical hypersurface.

Writing $h^1 = \sum_{p,q=1}^{m_1} \tilde{h}_{pq} \omega_p \otimes \omega_q$, by $C = 0$ and (2.3), we know h^1 is a Codazzi tensor on (M_1, g_1) , (4.27) means that there exists a space-like hypersurface

$$u : N^{m_1} \rightarrow \mathbb{S}_1^{m_1+1}(r) \subset \mathbb{R}_1^{m_1+2},$$

with h^1 as its second fundamental form. Clearly, u has at least two non-zero principal curvatures. According to (4.27) and (2.6), we can prove directly that u is of constant mean curvature H_1 and constant scalar curvature R_1 satisfying

$$H_1 = \frac{n\lambda}{m_1}, \quad R_1 = \frac{m_1(m_1-1)}{r^2} + \frac{n-1}{n} - n(n-1)\lambda^2.$$

Thus x is locally conformal equivalent to the hypersurfaces in Example 3.5.

Subcase 2.2. $2\mu + \lambda^2 < 0$, $2\nu + \lambda^2 > 0$.

Set $r = (2\nu + \lambda^2)^{-\frac{1}{2}}$, then $2\mu + \lambda^2 = -r^{-2}$, and (M_2, g_2) can be locally identified with $\mathbb{S}^{n-m_1}(r)$. Let $v : \mathbb{S}^{n-m_1}(r) \rightarrow \mathbb{R}^{n-m_1+1}$ be the standard totally umbilical hypersurface.

Writing $h^1 = \sum_{p,q=1}^{m_1} \tilde{h}_{pq} \omega_p \otimes \omega_q$, by $C = 0$ and (2.3), we know h^1 is a Codazzi tensor on (M_1, g_1) , (4.27) means that there exists a space-like hypersurface

$$u : N^{m_1} \rightarrow \mathbb{H}_1^{m_1+1}(-r) \subset \mathbb{R}_2^{m_1+2},$$

with h^1 as its second fundamental form. Clearly, u has at least two non-zero principal curvatures. According to (4.27) and (2.6), we can prove directly that u is of constant mean curvature H_1 and constant scalar curvature R_1 satisfying

$$H_1 = \frac{n\lambda}{m_1}, \quad R_1 = -\frac{m_1(m_1-1)}{r^2} + \frac{n-1}{n} - n(n-1)\lambda^2.$$

Thus x is locally conformal equivalent to the hypersurfaces in Example 3.6.

Thus we complete the proof of Proposition 4.3 □

Using Proposition 4.1, Proposition 4.2 and Proposition 4.3, we finish the proof of Theorem 1.2.

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